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DOUBLY TWISTED PRODUCT CONTACT CR-SUBMANIFOLDS IN KENMOTSU MANIFOLDS

MEHMET ATÇEKEN

Department of Mathematics
Faculty of Arts and Sciences
Gaziosmanpasa University
Tokat, 60250
Turkey
e-mail: matceken@gop.edu.tr

Abstract

In the present article, we have investigated the doubly twisted product contact CR-submanifolds in Kenmotsu manifolds. Finally, we see that there do not exist the proper doubly twisted product contact CR-submanifolds in Kenmotsu manifolds such that covariant vector field is tangent to submanifold.

1. Introduction

Twisted and warped products are widely used in geometry as well as physics. These manifolds have application fields in geometry and physics. Twisted product metric tensors, as a generalization of warped product metric tensor, have also been useful in the study of several aspects of submanifold theory. The relations between the twisted and warped product structures in semi-Riemannian geometry are researched [3, 10].

Recently, authors showed in [6] that there do not exist warped product contact CR-submanifolds of trans-Sasakian manifolds. In the

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present article, we have extended this study to the doubly twisted product contact CR-submanifolds in Kenmotsu manifold, which are an important class of manifolds. Because the existence or non-existence of these submanifolds are very important.

The geometry of doubly warped (twisted) product submanifolds have been researched in various type manifolds by many authors (see references).

There essentially the following is proven.

Let (N_1, g_1) and (N_2, g_2) be a semi-Riemannian manifolds of dimensions n_1 and n_2, respectively, and $\pi : N_1 \times N_2 \to N_1$ and $\sigma : N_1 \times N_2 \to N_2$ be the canonical projections. Furthermore, let $b : N_1 \times N_2 \to (o, \infty)$ and $f : N_1 \times N_2 \to (o, \infty)$ be smooth functions. Then, the doubly twisted product of (N_1, g_1) and (N_2, g_2) with twisting functions b and f is defined to be the product manifold $N = N_1 \times N_2$ with metric tensor $g = f^2 g_1 \oplus b^2 g_2$ given by $g = f^2 \pi^* g_1 + b^2 \sigma^* g_2$. For brevity in notation, we denote this semi-Riemannian manifold (N, g) by ${}_f N_1 \times {}_b N_2$. In particular, if $f = 1$, then $N_1 \times {}_b N_2$ is called the *twisted product* of (N_1, g_1) and (N_2, g_2) with twisting function b. Moreover, if b only depends on the points of N_1, then $N_1 \times {}_b N_2$ is called the *warped product* of (N_1, g_1) and (N_2, g_2) with warping function b. As a generalization of the warped product of two semi-Riemannian manifolds, ${}_f N_1 \times {}_b N_2$. is called the *doubly warped product* of semi-Riemannian manifolds (N_1, g_1) and (N_2, g_2) with warping functions b and f, if b and f only depend on the points of N_1 and N_2, respectively [3].

Let g be a semi-Riemannian metric tensor on the manifold $N = N_1 \times N_2$ and assume that the canonical foliations $\Gamma(TN_1)$ and $\Gamma(TN_2)$ intersect perpendicularly everywhere. Then g is the metric tensor of:

(1) a doubly twisted product ${}_fN_1 \times {}_bN_2$, if and only if $\Gamma(TN_1)$ and $\Gamma(TN_2)$ are totally umbilical foliations,

(2) a twisted product $N_1 \times {}_bN_2$, if and only if $\Gamma(TN_1)$ is totally geodesic foliation and $\Gamma(TN_2)$ is totally umbilical foliation,

(3) a warped product $N_1 \times {}_bN_2$, if and only if $\Gamma(TN_1)$ is totally geodesic foliation and $\Gamma(TN_2)$ is spheric foliation,

(4) a usual product of semi-Riemannian manifolds, if and only if $\Gamma(TN_1)$ and $\Gamma(TN_2)$ are totally geodesic foliations.

Now, let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds with Levi-Civita connections ∇^1 and ∇^2, respectively, and ∇ both denote the Levi-Civita connection and the gradient of the doubly twisted product ${}_fN_1 \times {}_bN_2$ of (N_1, g_1) and (N_2, g_2) with twisting functions b and f. Then, we have the following Proposition from [3].

Proposition 1.1. *For any* $X, Y \in \Gamma(TN_1)$ *and* $V \in \Gamma(TN_2)$, *we have*

$$\nabla_X Y = \nabla^1_X Y + X\log(f)Y + Y\log(f)X - g(X, Y)\nabla\log(f),$$

$$\nabla_X V = V\log(f)X + X\log(b)V,$$

where $\Gamma(TN)$ *denote the set of differentiable vector fields on N.*

2. Preliminaries

Let M be a $(2m+1)$- dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g). This means φ is a (1, 1) tensor field, ξ is a vector field, η is a 1-form, and g is the Riemannian metric on M such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0, \tag{1}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2}$$

for any $X, Y \in \Gamma(TM)$. An almost contact metric manifold is called *Kenmotsu manifold*, if the derivative of φ satisfies

$$(\overline{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \overline{\nabla}_X \xi = X - \eta(X)\xi, \tag{3}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on M [4].

Let N be an immersed submanifold in Kenmotsu manifold M, and we also denote by g the induced metric tensor on N. Denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on M and N, respectively, then Gauss and Weingarten formulas are, respectively, given

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{5}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where $\nabla^{\perp}$ is the connection on $TN^{\perp}$, h is the second fundamental form of N in M, and A_V is the shape operator associated with V. The second fundamental form h and the shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), N). \tag{6}$$

Now, for any $X \in \Gamma(TN)$, we put

$$\varphi X = tX + \omega X, \tag{7}$$

where tX and ωX denote the tangential and normal components of φX, respectively. In the same way, for any $V \in \Gamma(TN^{\perp})$, we put

$$\varphi V = BV + CV, \tag{8}$$

where BV and CV denote the tangential and normal components of φV, respectively.

The submanifold N is said to be *invariant* (resp., anti invariant), if ω (resp., t) is identically zero. Furthermore, for submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds.

A submanifold N tangent to ξ is called *contact CR-submanifold*, if it admits an invariant distribution D, whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, i.e., $TN = D \oplus D^{\perp} \oplus < \xi >$ with $\varphi(D_p) = D_p$ and $\varphi(D_p^{\perp}) \subset TN^{\perp}$ for each $p \in N$.

If the manifolds N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of Kenmotsu manifold M, respectively, then their doubly twisted product in the form $N = {}_fN_T \times {}_bN_{\perp}$.

3. Doubly Twisted Product Contact CR-Submanifolds

Throughout this section, we assume that M is a Kenmotsu manifold and $N = {}_fN_T \times {}_bN_{\perp}$ be a doubly twisted product contact CR-submanifold of Kenmotsu manifold M. Such submanifolds are always tangent to the structure vector field ξ. Therefore, we are concern with two cases.

Case 1. ξ is tangent to N_T.

Case 2. ξ is tangent to $N_{\perp}$.

First, we start with Case 1.

Theorem 3.1. *Let M be a $(2m+1)$-dimensional Kenmotsu manifold. Then, there do not exist doubly twisted product contact CR-submanifolds in the form $N = {}_fN_T \times {}_bN_{\perp}$ in M such that ξ is tangent to N_T.*

Proof. We suppose that $N = {}_fN_T \times {}_bN_{\perp}$ is a doubly twisted product contact CR-submanifold of a Kenmotsu manifold M. By using (3), (8), and Proposition 1.1, we have

$$(\overline{\nabla}_U\varphi)X = \overline{\nabla}_U\varphi X - \varphi(\overline{\nabla}_U X),$$

$$-\eta(X)\varphi U = \varphi X\log(b)U + U\log(f)\varphi X + h(U, \varphi X) - X\log(b)\varphi U$$

$$- U\log(f)\varphi X - Bh(X, U) - Ch(X, U), \tag{9}$$

for any $X \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$. From the tangential and normal components of (9), respectively, we have

$$\varphi X\log(f)U = Bh(X, U), \tag{10}$$

and

$$h(U, \varphi X) = X\log(b)\varphi U - \eta(X)\varphi U + Ch(U, X). \tag{11}$$

In the same way, we have

$$(\overline{\nabla}_X\varphi)U = \overline{\nabla}_X\varphi U - \varphi(\overline{\nabla}_X U),$$

$$g(\varphi X, U)\xi - \eta(U)\varphi X = -A_{\varphi U}X + \nabla_X^{\perp}\varphi U - X\log(b)\varphi U$$

$$- U\log(f)\varphi X - Bh(X, U) - Ch(X, U), \tag{12}$$

for any $X \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$. From the tangential and normal components of (12), respectively, we get

$$-A_{\varphi U}X = U\log(f)\varphi X + Bh(X, U), \tag{13}$$

and

$$\nabla_X^{\perp}\varphi U = X\log(b)\varphi U + Ch(X, U). \tag{14}$$

On the other hand, from (10), we have

$$g(Bh(X, U), X) = 0. \tag{15}$$

Moreover, from (6), (13), and (15), we get

$$-g(A_{\varphi U}X, X) = g(U\log(f)\varphi X + Bh(X, U), X),$$

$$-g(h(X, X), \varphi U) = U\log(f)g(\varphi X, X) + g(Bh(X, U), X) = 0. \tag{16}$$

Taking account of h being symmetric and bilinear, we get

$$g(h(X, Y), \varphi U) = 0, \tag{17}$$

for any $X, Y \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$. Thus making use of (13), (15), and (17), we conclude

$$U \log(f) g(\varphi X, \varphi Y) = -g(A_{\varphi U} X, \varphi Y) - g(Bh(X, U), \varphi Y)$$

$$= -g(h(X, \varphi Y), \varphi U) = 0. \tag{18}$$

Since g is a Riemannian metric, φX and φY are a non-null vectors, we conclude that f is constant on $N_\perp$.

On the other hand, considering (4), (5), and Proposition 1.1, we have

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi),$$

$$X - \eta(X)\xi = \nabla^T_X \xi + X \log(f)\xi + \xi \log(f) X - \eta(X) \nabla \log(f)$$

$$+ h(X, \xi), \tag{19}$$

where ∇^T denote the Levi-Civita connection on N_T. Multiplicating both sides of (19) by ξ, we conclude

$$g(X, \xi) - \eta(X) g(\xi, \xi) = g(\nabla^T_X \xi, \xi) + X \log(f) + \xi \log(f) \eta(X)$$

$$- \eta(X) g(\nabla \log(f), \xi),$$

$$0 = X \log(f), \tag{20}$$

that is, f is constant on N_T, which proves our assertion. ∎

Theorem 3.1 tells us that there exist no doubly twisted product contact CR-submanifolds in the form $N = {}_f N_T \times {}_b N_\perp$ in Kenmotsu manifolds such that ξ is tangent to invariant submanifold N_T. Next, we will research the existence or non-existence of doubly twisted product contact CR-submanifolds in the form $N = {}_f N_T \times {}_b N_\perp$ in Kenmotsu manifolds such that ξ is tangent to invariant submanifold $N_\perp$.

Theorem 3.2. *Let M be a* $(2m+1)$-*dimensional Kenmotsu manifold. Then, there do not exist doubly twisted product contact CR-submanifolds in the form* $N = {}_f N_T \times {}_b N_\perp$ *in M such that* ξ *is tangent to* $N_\perp$.

Proof. We suppose that $N = {}_fN_T \times {}_bN_\perp$ is a twisted product contact CR-submanifold of a Kenmotsu manifold M. Then by using (3), (4), and consider Proposition 1.1, we have

$$\overline{\nabla}_X\xi = \nabla_X\xi + h(X, \xi),$$

$$X - \eta(X)\xi = X\log(b)\xi + \xi\log(f)X + h(X, \xi), \tag{21}$$

for any $X \in \Gamma(TN_T)$. By multiplicating both sides of (21) by ξ, we get $X\log(b) = 0$, that is, b is constant on N_T. In the same way, we have

$$\overline{\nabla}_U\xi = \nabla_U\xi + h(U, \xi),$$

$$\begin{aligned} U - \eta(U)\xi = \nabla^\perp_U\xi + U\log(b)\xi + \xi\log(b)U - g(\xi, U)\nabla\log(b) \\ + h(U, \xi), \end{aligned} \tag{22}$$

for any $U \in \Gamma(TN_\perp)$, where $\nabla^\perp$ denote the Levi-Civita connection on $N_\perp$. Also, multiplicating both sides of (22) by ξ, we derive

$$\eta(U) - \eta(U) = U\log(b) + \eta(U)\xi\log(b) - \eta(U)\xi\log(b)$$

$$0 = U\log(b), \tag{23}$$

which implies that b is constant on $N_\perp$. ■

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